[07-01-04-T12C]

4-8

THE DEFINITE INTEGRAL AND THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

In Articles 4–5, 4–6, and 4–7, we made a systematic study of the area problem. We have arrived at the following result.

If the function f is positive and continuous over the domain $a \le x \le b$, then the area under its graph is

$$A_a^b = \lim \sum f(c_k)\,\Delta x = \int f(x)\,dx\Big]_a^b = F(x)\Big]_a^b = F(b) - F(a). \tag{1}$$

The first part of this equation is just the definition of the area as the limit of the sum of areas of inscribed rectangles. The last part of the equation gives a short way to evaluate this limit by calculus. Therein lies one of the most powerful ideas of post-Renaissance mathematics, for the key idea is this: The limit can be evaluated by integration. This, essentially, is what is known as the *Fundamental Theorem* of integral calculus. It ties together the summation process (which Archimedes used over two thousand years ago for finding areas, volumes, and centers of gravity) and the differentiation process, from which one may find the tangent to a curve. It is a remarkable fact that the inverse of the "tangent problem" (that is, the inverse of differentiation) provides a ready tool for solving the summation problem. And its applications, as we shall see, extend far beyond the finding of areas; e.g., finding volumes of solids, lengths of curves, areas of surfaces of revolution, centers of gravity, work done by a variable force, gravitational and electrical potential, population growth, and cardiac output, to mention only a few.

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While Eq. (1) above is expressed in terms of area, and up until now has been restricted to positive-valued functions, the Fundamental Theorem is less restrictive.

Fundamental Theorem of Integral Calculus. *Let f be a function that is continuous over the domain $a \le x \le b$. Let*

$$a,\quad x_1,\quad x_2,\quad \ldots,\quad x_{n-1},\quad b \tag{2}$$

be a set of numbers $a < x_1 < x_2 < \cdots < x_{n-1} < b$, which partition the interval (a, b) into n equal subintervals each of length

$$\Delta x = \frac{(b-a)}{n}. \tag{3}$$

Let $c_1, c_2, \ldots, c_n$ be a set of n numbers, one in each subinterval,

$$a \le c_1 \le x_1, \qquad x_1 \le c_2 \le x_2, \qquad \ldots, \qquad x_{n-1} \le c_n \le b. \tag{4}$$

Let

$$\begin{aligned} S_n &= f(c_1)\,\Delta x + f(c_2)\,\Delta x + \cdots + f(c_n)\,\Delta x \\ &= \sum_{k=1}^{n} f(c_k)\,\Delta x. \end{aligned} \tag{5}$$

Finally, let $F(x)$ be any integral of $f(x)\,dx$,

$$F(x) = \int f(x)\,dx. \tag{6}$$

Then, as $n \to \infty$,

$$\lim S_n = \lim \sum f(c_k)\,\Delta x = F(b) - F(a). \tag{7}$$

Proof. We shall first prove (7) for the special set of numbers $c_1, c_2, \ldots, c_n$ that we get by applying the Mean Value Theorem of Article 3–8 to the function F in each subinterval. We can do this because F is differentiable and continuous. Thus, remembering that $F'(x) = f(x)$, we have

$$\begin{aligned}
F(x_1) - F(a) &= F'(c_1) \cdot (x_1 - a) = f(c_1)\,\Delta x,\\
F(x_2) - F(x_1) &= F'(c_2) \cdot (x_2 - x_1) = f(c_2)\,\Delta x,\\
F(x_3) - F(x_2) &= F'(c_3) \cdot (x_3 - x_2) = f(c_3)\,\Delta x,\\
&\vdots\\
F(x_{n-1}) - F(x_{n-2}) &= F'(c_{n-1}) \cdot (x_{n-1} - x_{n-2}) = f(c_{n-1})\,\Delta x,\\
F(b) - F(x_{n-1}) &= F'(c_n) \cdot (b - x_{n-1}) = f(c_n)\,\Delta x.
\end{aligned} \tag{8}$$

We add Eqs. (8), and note that $F(x_1), F(x_2), \ldots, F(x_{n-1})$ all appear twice on the left side, once positive and once negative. Hence these terms cancel out, leaving only $F(b) - F(a)$ in the sum. Thus we get

$$F(b) - F(a) = f(c_1)\,\Delta x + f(c_2)\,\Delta x + \cdots + f(c_n)\,\Delta x. \tag{9}$$

Since the left side of this equation does not in any way involve n, it remains fixed as we let $n \to \infty$, thus establishing Eq. (7), for this particular way of choosing the numbers $c_1, c_2, \ldots, c_n$.

But the theorem states that the same answer is obtained no matter how the c's are chosen in the subintervals, so long as there is one c in each subinterval. To establish this final result, we recall that the function f is continuous on the closed interval $a \le x \le b$, and therefore is *uniformly* continuous there [Article 2–11, Theorem 5]. Hence, if ϵ is any positive number, there exists a positive number δ, depending only upon ϵ, such that

$$|f(c_k) - f(c_k')| < \epsilon \tag{10}$$

whenever

$$|c_k - c_k'| < \delta. \tag{11}$$

And we can make $\Delta x = (b - a)/n < \delta$ by making

$$n > \frac{(b - a)}{\delta}. \tag{12}$$

For all sufficiently large n, (12) is satisfied. Now let c_1, c_1' be two numbers in the first subinterval, c_2, c_2' in the second, and so on. Form the sums

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x,$$

$$S_n' = \sum_{k=1}^{n} f(c_k')\,\Delta x.$$

Their difference is less than or equal to

$$|f(c_1)-f(c_1')|\,\Delta x + |f(c_2)-f(c_2')|\,\Delta x + \cdots + |f(c_n)-f(c_n')|\,\Delta x. \tag{13}$$

Every term in (13) is less than $\epsilon \cdot \Delta x$, by (10), and there are n terms. Therefore

$$|S_n - S_n'| < n \cdot (\epsilon \cdot \Delta x) = \epsilon \cdot (n\,\Delta x) = \epsilon \cdot (b-a), \tag{14}$$

provided condition (12) is satisfied. This inequality, (14), says that the sums S_n and S_n' that we get from two different choices of the c's in the subintervals can be made to differ by as little as we please [no more than $\epsilon \cdot (b-a)$] by making n sufficiently large. But for the particular choice of the c's in Eqs. (8), we have

$$S_n = F(b) - F(a).$$

Therefore S_n' differs arbitrarily little from $F(b) - F(a)$ when n is sufficiently large. This means that

$$\lim S_n' = F(b) - F(a). \tag{15}$$

Equation (15) completes the proof.

REMARK. The integral sign, $\int$, is a modified capital S (for sum), intended to remind us of the close connection between integration and summation.

The limit

$$\lim \sum f(c_k)\,\Delta x$$

in Eq. (7) is called the *definite integral* of *f from a to b*. It is denoted by the symbol

$$\int_a^b f(x)\,dx.$$

The numbers a and b are called the *limits of integration* of the integral, a being the *lower limit* and b the *upper limit*.

The Fundamental Theorem tells us that we can evaluate a definite integral if we know any indefinite integral $F(x)$ of the integrand $f(x)\,dx$. We just subtract $F(a)$ from $F(b)$.

EXAMPLE 1

$$\begin{aligned}\int_0^{\pi/2} \cos x\,dx &= \sin x\Big]_0^{\pi/2}\\ &= \sin\frac{\pi}{2} - \sin 0\\ &= 1.\end{aligned}$$